Int. J. Solids Structures, 1978, Vol. 14, pp. 53–65. Pergamon Press. Printed in Great Britain

DERIVATION OF GEOMETRIC STIFFNESS MATRIX FOR FINITE ELEMENT HYBRID DISPLACEMENT MODELS

KLAUS BRANDT
Fachgebiet Leichtbau, Technische Hochschule Darmstadt, Petersenstraße 18, West Germany

(*Received* 13 *February* 1976; *revised* 19 *April* 1977; *received for publication* 16 *July* 1977)

Abstract—The matrix power series expansion is used for deriving the static geometric stiffness and elastic stiffness matrix of a hybrid displacement model. No restrictions about the number of interpolation functions within the interior of the element are introduced. The generalized degrees-of-freedom are not defined on nodal points but in an abstract way on element boundaries. Plate bending problems are considered to demonstrate the method.

INTRODUCTION

The original concept of the hybrid stress finite element model based on the complementary energy principle and proposed by Pian[1] 1964 and the assumed displacement hybrid model initiated by Tong[2] 1970 (Jones'[3] earlier formulation may be interpreted as the first hybrid deformation model) were limited to static analysis and have been later extended to dynamic and buckling problems (e.g. Allman[4], Kikuchi and Ando[5]). Tong *et al.*[6] presented a unifying approach in deriving the static geometric stiffness and mass matrices for finite element hybrid models. The formulation is based on a modified Reissner variation principle. In order to derive these matrices it is assumed that the number of interpolation functions (this means the number of degrees-of-freedom) within the element is equal to that of the generalized degrees-of-freedom on the element boundaries which is also the usual way in the common finite element method. In computing the natural frequencies the author[7] has demonstrated that this restriction can be dropped. Both, the generalized potential energy principle and the generalized complementary energy principle have been used to derive the static element mass and stiffness matrices. For plate bending problems the natural frequencies have been computed and the results from both finite element hybrid models have been compared.

The purpose of this paper is to show that for initial stress problems the exact solution of the discretized element equations derived on the basis of a generalized potential energy principle leads to a generalized stiffness matrix whereas the approximate solution results in a formulation using the elastic stiffness and geometric stiffness matrices or in the case of buckling problems in an eigenvalue matrix equation of the conventional buckling analysis. It will be demonstrated that these matrices may be obtained from a matrix power series expansion. No restrictions about the number of interpolation functions (apart from necessary conditions with regard to the number of generalized element degrees-of-freedom) are introduced. Contrary to the common procedure the element degrees-of-freedom are not defined on nodal points but in an abstract way at element boundaries according to [3] and [9]. In the text a Cartesian coordinates system is used.

THE GENERALIZED VARIATIONAL PRINCIPLE

Taking into account the effect of initial in-plane stresses the functional may be written in the following form

$$\pi(v_i; p_i) = 1/2 \int_I \sum_{ijkl} C_{ijkl} v_{i,j} v_{k,l} \, dI + 1/2 \int_I \sum_{kij} \sigma^0_{ij} v_{k,i} v_{k,j} \, dI - \sum_i \int_S p_i (v_i - u_i) \, dS = \text{stationary} \tag{1}$$

where I is the volume of the region, S is the boundary of I (in the sense of the finite element method I refers to the element volume and S to the element boundary), σ_{ij} is the stress tensor, σ^0_{ij} is the initial stress tensor, v_i are the displacements and ϵ_{ij} is the strain tensor in I, p_i and u_i are the boundary tractions and displacements on S, C_{ijkl} are the elastic constants and n_j are the components of the unit vector normal to the boundary (Fig. 1). By initial stresses, we mean

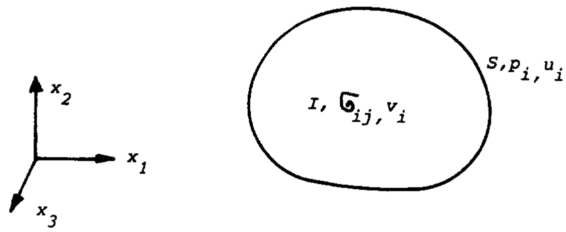

Fig. 1.

those stresses which have existed in a body in the initial state, that is, before the start of deformation of interest. The initial state is chosen as the reference state of an initial stress problem.

The independent quantities subject to variation in the functional (1) are the displacements v_i in the region I and the boundary tractions p_i. The displacements u_i on the boundary S should not be varied in this principle. The body forces are neglected and it is not specified in the functional whether the boundary displacements or the boundary tractions are prescribed. The reason is that in the theory of the finite element method the discretized quations are derived by disregarding the boundary conditions and this circumstances should also be accommodated in the mathematical formulation.

The variation with respect to v_i and p_i and the subsidiary conditions

$$\epsilon_{ij} = 1/2\,(v_{i,j} + v_{j,i}) \tag{2}$$

$$\sigma_{ij} = \sum_{kl} C_{ijkl}\epsilon_{kl} \tag{3}$$

yield

$$\delta\pi(v_i; p_i) = -\int_I \sum_k \left(\left(\sum_i \sigma^0_{ij} v_{k,i}\right)_{,j} + \sum_j \sigma_{kj,j}\right)\delta v_k \,\mathrm{d}I + \sum_k \int_s \left(\sum_{ij} \sigma^0_{ij} n_j v_{k,i} + \sum_j \sigma_{kj} n_j\right)\delta v_k \,\mathrm{d}S - \sum_i \int_s p_i \delta v_i \,\mathrm{d}S - \int_S \sum_i (v_i - u_i)\,\delta p_i \,\mathrm{d}S. \tag{4}$$

The vanishing $\delta\pi(v_i; p_i)$ for an arbitrary δv_k in I, and an arbitrary δp_i on the boundary will give the equations:

(1) the equilibrium equation in I

$$\sum_j \left(\left(\sum_i \sigma^0_{ij} v_{k,i}\right)_{,j} + \sigma_{kj,j}\right) = 0 \tag{5}$$

(2) the kinematic boundary conditions on S

$$v_k - u_k = 0 \tag{6}$$

(3) the boundary equilibriun conditions on S

$$\sum_{ij} \sigma^0_{ij} n_j v_{k,i} + \sum_j \sigma_{kj} n_j - p_k = 0. \tag{7}$$

STRUCTURES COMPOSED OF ELEMENTS

Now we consider a structure composed of elements (Fig. 2). The interior parts of elements are I_E and the element boundaries s_{Er} while S_R denotes the boundary parts of the structure.

The topological behaviour is described by a coincidence condition

$$s_{Er} = \sum_R B_{ErR} S_R \tag{8}$$

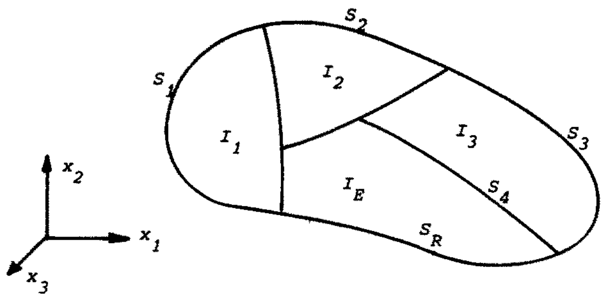

Fig. 2. Assembled structure.

and the continuity and equilibrium conditions are given by

$$u_{Eri} = \sum_R T_{ErR} U_{ri} \tag{9}$$

$$P_{Ri} = \sum_{Er} T_{ErR} p_{Eri} \tag{10}$$

where U_{Ri} and P_{Ri} are surface displacements and surface tractions of the assembled structure, u_{Eri} and p_{Eri} are surface displacements and surface tractions of the elements boundaries r, E indicates the elements and T_{ErR} is a transformation matrix.

The surface tractions on the structure boundaries are approximated by interpolation functions

$$P_{Ri} = \sum_n Y_{ni}(S_R)\tilde{P}_{Rn} \tag{11}$$

and for the element boundaries

$$p_{Eri} = \sum_n \varphi_{ni}(s_{Er})\tilde{p}_{Em} \tag{12}$$

where $\tilde{P}_{Rn}$ and $\tilde{p}_{Em}$ are unknown coefficients. Such interpolation functions are chosen, which are associated with the single boundary parts. Making use of eqns (8)–(10) we obtain

$$Y_{ni}(S_R) = \varphi_{ni}(s_{Er})$$

and the analogous form of the continuity and the equilibrium conditions for the discretized quantities

$$\tilde{u}_{Em} = \sum_R T_{ErR}\tilde{U}_{Rn} \tag{13}$$

$$\tilde{P}_{Rn} = \sum_{Er} T_{ErR}\tilde{p}_{Em}. \tag{14}$$

The generalized surface displacements are defined by

$$\tilde{U}_{Rn} = \sum_i \int_S Y_{ni}(S_R)U_{Ri}\,\mathrm{d}S_R. \tag{15}$$

On the base of the theory formulated above it is possible to derive finite element approximations in a general form and a separate treatment of the assembled structure and the elements is guaranteed (contrary to the formulation of the finite element method based on a generalized variational principle as used by Greene and his associates[8]).

THE FORMULATION OF THE HYBRID FINITE ELEMENT DISPLACEMENT MODEL

The variational theorem presented in the preceding section may be used to derive finite element models. To construct the finite element equations the displacements within the element and the surface tractions are approximated by interpolation functions:

$$v_{Ei}(x_j) = \sum_{\alpha} \tilde{v}_{E\alpha}\psi_{Ei\alpha}(x_j) \tag{16}$$

$$p_{Eri} = \sum_{\gamma} \tilde{p}_{Er\gamma}\varphi_{Ei\gamma}(s) \tag{17}$$

where $\tilde{v}_{E\alpha}$ and $\tilde{p}_{Er\gamma}$ are unknown coefficients. By varying the functional with respect to $\tilde{p}_{Er\gamma}$ and $\tilde{v}_{E\alpha}$ we obtain the element equations

$$\sum_{\beta} V_{E\alpha\beta}\tilde{v}_{E\beta} + \sum_{\beta} H_{E\alpha\beta}\tilde{v}_{E\beta} - \sum_{\gamma r} L_{Er\gamma}\tilde{p}_{Er\gamma} = 0 \tag{18}$$

$$-\sum_{\beta} L_{Er\beta\gamma}\tilde{v}_{E\beta} = -\tilde{u}_{Er\gamma} \tag{19}$$

with the strain energy matrix

$$V_{E\alpha\beta} = \int_{I_E} \sum_{ijkl} C_{ijkl}\psi_{Ei\alpha,j}\psi_{Ek\beta,l}\,\mathrm{d}I_E \tag{20}$$

the initial stress energy matrix

$$H_{E\alpha\beta} = \int_{I_E} \sum_{kij} \sigma_{ij}^{0}\psi_{Ek\alpha,i}\psi_{Ek\beta,j}\,\mathrm{d}I_E \tag{21}$$

the boundary energy matrix

$$L_{Er\alpha\gamma} = \int_{s_{Er}} \sum_{i} \psi_{Ei\alpha}\varphi_{Ei\gamma}\,\mathrm{d}s_{Er} \tag{22}$$

and the definition of the generalized displacements

$$\tilde{u}_{Er\gamma} = \int_{s_{Er}} \sum_{i} u_{Eri}\varphi_{Ei\gamma}\,\mathrm{d}s_{Er}. \tag{23}$$

The generalized surface displacements are related to physical surfaces displacements by an integral while the corresponding boundary tractions are defined by an interpolation formula[9].

THE SOLUTION OF THE DISCRETIZED ELEMENT EQUATIONS

For solving the equations in the sense of the finite element procedure the $\tilde{v}_{E\alpha}$ have to be expressed in terms of $\tilde{u}_{Er\gamma}$. Pian and his associates[6] have proposed to choose the number of interpolation functions within the element equal to that of the generalized element degrees-of-freedom. This means $[L]_E$ is a square matrix and we obtain from eqn (19)

$$\{\tilde{v}\}_E = [L]_E^{-1}\{\tilde{u}\} \tag{24}$$

the following equation

$$[L]_E^{-T}[V]_E[L]_E^{-1}\{\tilde{u}\}_E + [L]_E^{-T}[H]_E[L]_E^{-1}\{\tilde{u}\}_E = \{\tilde{p}\}_E \tag{25}$$

or

$$[K_0]_E\{\bar{u}\}_E + [G_0]_E\{\bar{u}\}_E = \{\bar{p}\}_E \tag{26}$$

where $[K_0]_E$ is the elastic element stiffness matrix and $[G_0]_E$ the element geometric stiffness matrix.

In the case of a buckling problem the initial stresses σ_{ij}^0 are expressed in terms of a single loading parameter $\sigma_{ij}^0 = \lambda\bar{\sigma}_{ij}^0$ and hence eqn (26) can be written

$$[K_0]_E\{\bar{u}\}_E + \lambda[\bar{G}_0]_E\{\bar{u}\}_E = \{\bar{p}\}_E. \tag{27}$$

In general, if more degrees-of-freedom are used in the interior as at the element boundaries, the matrix $[L]_E$ will be a rectangular matrix and the $\bar{v}_{E\alpha}$ cannot be expressed by using eqn (19). In this case two solution procedures of the discretized element equations are possible: The exact solution leads to a (generalized) stiffness matrix while an approximate solution results in a formulation using the elastic stiffness matrix and a geometric stiffness matrix or in the case of buckling problems in a linear eigenvalue form.

First we consider the exact solution. The simplest way to construct the stiffness matrix is to invert the global matrix in eqns (18) and (19).

$$\begin{bmatrix} -[V]-[H] & [L] \\ [L]^T & [O] \end{bmatrix}_E^{-1} = \begin{bmatrix} [A] & [R] \\ [R]^T & [K_s] \end{bmatrix}_E. \tag{28}$$

Then we obtain

$$\{\bar{v}\}_E = [R]_E\{\bar{u}\}_E \tag{29}$$

$$\{\bar{p}\}_E = [K_s]_E\{\bar{u}\}_E \tag{30}$$

where $[K_s]_E$ is the (generalized) element stiffness matrix. It should be remarked that the matrix $[K_S]_E$ reduces to the elastic element stiffness matrix $[K_0]_E$ when initial stresses are zero.

For buckling problems the (generalized) element stiffness matrix is a function of the unknown parameter λ. When a transformation is introduced to relate the displacements $\{\bar{u}\}_E$ and the boundary tractions $\{\bar{p}\}_E$ to the independent global displacements $\{\tilde{U}\}$ and global forces $\{\tilde{P}\}$ the condition to determine the buckling load is given by

$$\det[K_s] = 0 \tag{31}$$

where $[K_s]$ is obtained by assembling the matrices of the individual elements. Obviously, the formulation leads to a determinantal equation and not to the conventional eigenvalue problem. The factor λ is found by a systematic search of the zeros of the determinant which means that the factor has to be estimated for constructing the (generalized) element stiffness matrix in each step until the zeros of the determinant have been found. Link[10] has used this procedure in connection with a hybrid displacement finite element model. To overcome these difficulties we consider an approximate solution of the discretized element equations.

From eqns (18 and (19) in the following form

$$\begin{bmatrix} -[V] & [L] \\ [L]^T & [0] \end{bmatrix}_E \begin{Bmatrix} \{\bar{v}\} \\ \{\bar{p}\} \end{Bmatrix}_E = \begin{bmatrix} [H] & [O] \\ [O] & [I] \end{bmatrix}_E \begin{Bmatrix} \{\bar{v}\} \\ \{\bar{u}\} \end{Bmatrix}_E \tag{32}$$

we obtain by using

$$\begin{bmatrix} -[V] & [L] \\ [L]^T & [O] \end{bmatrix}_E^{-1} = \begin{bmatrix} -[B] & [C] \\ [C]^T & [K_0] \end{bmatrix}_E \tag{33}$$

$$\{\bar{v}\}_E + [B]_E[H]_E\{\bar{v}\}_E = [C]_E\{\bar{u}\}_E \tag{34}$$

$$\{\bar{p}\}_E = [C]_E^T[H]_E\{\bar{v}\}_E + [K_0]_E\{\bar{u}\}_E \tag{35}$$

where $[I]_E$ is the unit matrix and $[K_0]_e$ is the elastic element stiffness matrix. Equation (34) yields

$$\{\tilde{v}\}_E = [[I]_E + [B]_E[H]_E]^{-1}[C]_E\{\tilde{u}\}_E. \tag{36}$$

Substituting this expression for $\{\tilde{u}\}_E$ into eqn (35) we find

$$[K_0]_E\{\tilde{u}\}_E + [C]_E{}^T[H]_E[[I]_E + [B]_E[H]_E]^{-1}[C]_E\{\tilde{u}\}_E = \{\tilde{p}\}_E. \tag{37}$$

The expansion of the second term in eqn (37) in series form

$$[[I]_E + [B]_E[H]_E]^{-1} = [I]_E - [C]_E[H]_E + [C]_E[H]_E[C]_E[H]_E - \cdots \tag{38}$$

yields

$$[K_0]_E\{\tilde{u}\}_E + [G_0]_E\{\tilde{u}\}_E - [D_0]\{\tilde{u}\}_E + \cdots = \{\tilde{p}\}_E \tag{39}$$

or in the case of buckling problems we find

$$[K_0]_E\{\tilde{u}\}_E + \lambda[\bar{G}_0]_E\{\tilde{u}\}_E - \lambda^2[\bar{D}_0]_E\{\tilde{u}\}_E + \cdots = \{\tilde{p}\}_E. \tag{40}$$

If all the terms with degrees higher than λ in eqn (40) or the corresponding matrices in (39) are omitted we obtain the following element equation

$$[K_0]_E\{\tilde{u}\}_E + \lambda[\bar{G}_0]_E\{\tilde{u}\}_E = \{\tilde{p}\}_E \tag{41}$$

or

$$[K_0]_E\{\tilde{u}\}_E + [G_0]_E\{\tilde{u}\}_E = \{\tilde{p}\}_E \tag{42}$$

respectively where

$$[G_0]_E = [C]_E{}^T[H]_E[C]_E$$

is the element geometric stiffness matrix, while the remaining matrices in eqn (39) are higher order geometric matrices.

The transformation to the independent global boundary displacements and tractions leads in the case of buckling problems to the linear eigenvalue form

$$[K_0]\{U\} + \lambda[\bar{G}_0]\{U\} = 0 \tag{43}$$

which can be solved by a standard eigenvalue routine.

BUCKLING OF PLATES

The theory developed in the former section has been used to analyse the buckling of plates. Rectangular elements are chosen for studying the buckling behaviour. The variational functional for plate problems corresponding to eqn (1) can be written as

$$\pi(w; V_r; M_r; Q_m) = 1/2\int_I \sum_{ijkl} C_{ijkl}w_{,ij}w_{,kl}\,dI + 1/2\int_I \sum_{ij} N_{ij}^0 w_{,j}w_{,i}\,dI - \sum_r \int_{s_r}(w_{,n} - \hat{w}_{r,n})M_r\,ds_r - \sum_r \int_{s_r}(w - \hat{w}_r)V_r\,ds_r - \sum_m (w - \hat{w}_m)Q_m = \text{stationary} \tag{44}$$

where:

w normal deflection within the region of the plate
m_{ij} moment tensor

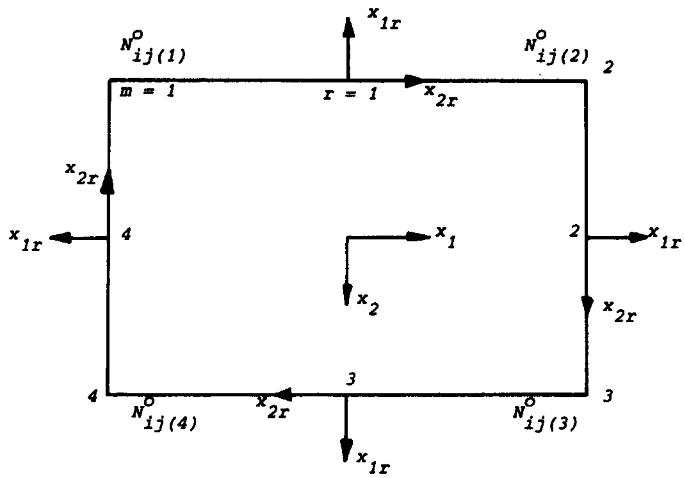

Fig. 3. Co-ordinate systems and element level numbering.

$\hat{w}_r$	displacement on boundary r
$\hat{w}_{r,n}$	slope on boundary r
$\hat{w}_m$	corner displacements
n_{rj}	components of unit vector normal to boundary r
t_{rj}	components of unit tangential vector on boundary r
κ_{ij}	bending and twisting curvatures
$(\partial/\partial\bar{x}_1) \equiv {}_{,n}$;	$\dfrac{\partial}{\partial\bar{x}_2} \equiv {}_{,s}$; $\dfrac{\partial}{\partial x_i} \equiv {}_{,i}$; $\dfrac{\partial^2}{\partial x_i \partial x_j} \equiv {}_{,ij}$
V_r	effective shear forces on boundary r
M_r	bending moment on boundary r
Q_m	corner forces
N^0_{ij}	initial midplane stress resultants

The independent quantities subject to variation in the functional (44) are the displacement w, the effective shear forces V_r, the bending moment M_r and the corner forces Q_m. The modifiier $\wedge$ indicates that this quantity will not be varied. By means of

$$m_{ij} = -\sum_{kl} C_{ijkl} w_{,kl} \tag{45}$$

$$\kappa_{ij} = -w_{,ij} \tag{46}$$

the functional replaces the following equations:

(1) The equilibrium equation

$$\sum_{ij} m_{ij,ij} + \sum_{ij} N^0_{ij} w_{,ij} = 0. \tag{47}$$

(2) The kinematic boundary conditions

$$w - \hat{w}_r = 0 \tag{48}$$

$$w_{,n} - \hat{w}_{r,n} = 0. \tag{49}$$

(3) The kinematic corner conditions

$$\hat{w}_m - w = 0. \tag{50}$$

(4) The equilibrium boundary conditions

$$V_r - \sum_{ij} m_{ij,j}n_{ri} - \sum_{ij} (m_{ij}, n_{rj}t_{ri})_{,s} - \sum_{ij} N_{ij}^0 w_{,j}n_i = 0 \tag{51}$$

$$M_r + \sum_{ij} m_{ij}n_{ri}n_{rj} = 0. \tag{52}$$

(5) The equilibrium corner conditions

$$Q_m + \sum_{ij} m_{ij}n_{rj}t_{ri}\Big|_r^{r-1} = 0 \tag{53}$$

$|_r^{r-1}$ stands for contributions from sides r and $r-1$ which intersect at the corner m.

The corner point forces Q_m are included because the w displacement functions do not provide for automatic continuity of w at the corners, which is necessary in order to obtain as natural condition the equilibrium of the forces at the corner points which are present in the Kirchhoff bending of plates.

In constructing the element equations the displacement within the element is assumed to be (Fig. 3)

$$w_E = \sum_{n_1=0} \sum_{n_2=0} \tilde{w}_{En_1n_2}(x_1)^{n_1}(x_2)^{n_2}. \tag{54}$$

Thus the displacement within the elements is approximated using a complete n_{EI}-order polynomial sequence in terms of the local coordinates x_1, x_2. The moment M_r and the effective shear forces V_r at the boundaries are approximated by the following interpolation functions (see Fig. 3)

$$M_r = \sum_{g=0}^{n_{M_r}} \tilde{M}_{rg}(x_{2r})^g \tag{55}$$

$$V_r = \sum_{q=0}^{n_{Vr}} \tilde{V}_{rq}(x_{2r})^q. \tag{56}$$

The corner forces Q_m don't need to be discretized since they are already discrete variables.

The generalized surface displacements and surface slopes are

$$\tilde{w}_{qEr} = \int_{s_{Er}} \hat{w}_r x_{2\,r}^{\,q}\, ds_{E_r} \tag{57}$$

$$\tilde{w}_{gEr,n} = \int_{s_{Er}} \hat{w}_{r,n} x_{2\,r}^{\,g}\, ds_{Er} \tag{58}$$

and the discrete corner displacements

$$\tilde{w}_m = \hat{w}_m. \tag{59}$$

Using the interpolation functions in eqns (55) and (56) the generalized boundary displacements and the generalized boundary slopes in eqns (57) and 58) can be interpreted as the area (first term $\tilde{w}_{0Er}$ and $\tilde{w}_{0Er,n}$), as a moment of first order (second term $\tilde{w}_{1Er}$ and $\tilde{w}_{1Er,n}$) and as moments of higher order (the remaining terms in eqns (57) and (58)) of the physical displacements and physical slopes, respectively.

The initial midplane stress resultants within the element are assumed to be

$$\begin{aligned} N_{11}^0 &= C_0 + C_1x_1 + C_2x_2 + C_3x_1x_2 \\ N_{22}^0 &= C_4 + C_5x_1 + C_6x_2 + C_7x_1x_2 \\ N_{12}^0 &= -\left(C_1x_2 + \frac{C_3}{2}x_2^2 + C_5x_1 + \frac{C_7}{2}x_1^2 + C_8\right). \end{aligned} \tag{60}$$

By means of the equilibrium condition the coefficients C_j can be expressed in terms of stress resultants $N^0_{ij(m)}$ at nodal points.

NUMERICAL RESULTS

To establish the accuracy and the rate of convergence of the hybrid displacement model presented in the preceding section some simple test problems are computed. The examples refer to an isotropic simply supported square plate (length b, Poisson's ratio taken as $\nu = 0.3$) under uniform compression in one direction, pure shear, triangular loading (Fig. 4) and bending (Fig. 5). In all cases the buckling coefficient k is defined by

$$k = \frac{N^0_{ij} b^2}{\pi^2 D}$$

where D is the bending rigidity and N^0_{ij} the intensity of the initial midplane stress resultants of the corresponding loading case. For the hybrid displacement model three approximations have been used. First, a complete sixth-order polynomial for the normal deflection within the element and a complete first-order polynomial for the moment and the effective shear forces at the boundaries were assumed. This model is designated by 6.1.1. For the second model (7.2.2) a complete seventh-order polynomial in the interior and a second-order polynomial along the boundaries are used. The model 8.3.3 is derived from the assumption of an eighth degree displacement field within the element while the moment and the effective shear forces at the boundaries are assumed to be cubic functions. For example, the model 7.2.2 possess 36 degrees-of-freedom within the element, 3 degrees-of-freedom for the moment and 3 for the effective shear forces at each boundary. In this way the rectangular element has 24 generalized degrees-of-freedom at the boundaries to which 4 degrees-of-freedom at the corner points have to be added, increasing the total number of the generalized element degrees-of-freedom to 28.

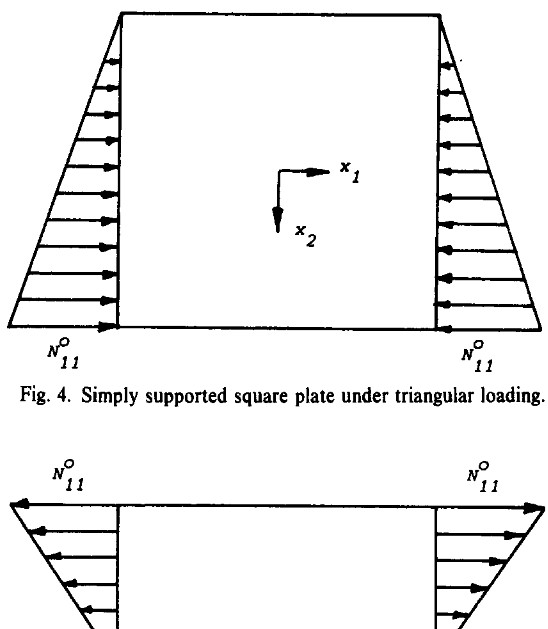

Fig. 4. Simply supported square plate under triangular loading.

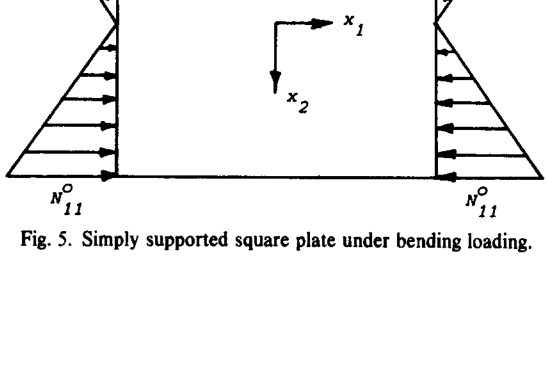

Fig. 5. Simply supported square plate under bending loading.

For studying the convergence behaviour of the buckling coefficient k dependent on the number of elements, the structure was divided into different regular meshes, namely 4, 9 and 16 square elements are chosen. The approximate solution of the discretized element equations is used (linear eigenvalue problem) to determine the buckling coefficients. The Tables 1–4 summarize the results. The values of the buckling coefficients are given in the first line, the number of total degrees-of-freedom of the entire structure in parenthesis and the percentage error in the third line.

The Tables 1–4 exhibit an interesting feature. It will be observed that the convergence in each case appears monotonic and it seems that the hybrid displacement model permits the determination of higher bounds for the buckling coefficients of the structure. The 7.2.2 model converged slightly more rapidly than the 6.1.1 and 8.3.3 model. This indicates for the chosen problems that the 6.1.1 and 8.3.3 model are more stiffened as 7.2.2.

Although all of these element models converge toward the exact solution further investigations of the buckling modes have shown that discontinuities in the normal displacement and in the bending slope between adjacent elements have occurred for the 6.1.1 model and only the 7.2.2 and 8.3.3 model have provided good results. Independent of the convergence behavior as observed here it should be emphasized that the variational principle in eqn (1) is not a minimum principle, which means that the solution can converge from above or from below toward the exact solution. Extensive numerical investigations[7] for dynamic problems with the

Table 1. Values of coefficient k for a simply supported square plate under uniform compression in one direction (exact value of $k = 4.0$, see Klöppel *et al.*[11])

Number of elements	Degree of polynomial		
	6.1.1	7.2.2	8.3.3
4	4.035 (33) 0,87 %	4.018 (49) 0,45 %	4.030 (65) 0,75 %
9	4.008 (76) 0,2 %	4.004 (112) 0,1 %	4.006 (148) 0,15 %

Table 2. Values of coefficient k for a simply supported square plate under pure shear (known value of $k = 9.34$, see [11])

Number of elements	Degree of polynomial		
	6.1.1	7.2.2	8.3.3
4	11.614 (33) 24,3 %	10.289 (49) 10,2 %	10.924 (65) 16,9 %
9	9.725 (76) 4,12 %	9.505 (112) 1,76 %	9.614 (148) 2,93 %
16	9.457 (137) 1,25 %	9.383 (201) 0,46 %	9.424 (265) 0,90 %

Table 3. Values of coefficient k for a simply supported square plate under triangular loading, see Fig. 2 (known value of $k = 7.81$, see [11])

Number of elements	Degree of polynomial		
	6.1.1	7.2.2	8.3.3
4	7.894 (33) 1,08 %	7.844 (49) 0,58 %	7.882 (65) 0,92 %
9	7.829 (76) 0,24 %	7.820 (112) 0,13 %	7.827 (148) 0,21 %
16	7.817 (137) 0,09 %	7.815 (201) 0,06 %	7.817 (265) 0,09 %

Table 4. Values of coefficient k for a simply supported square plate under bending loading, see Fig. 3 (known value of $k = 25.4$ see [11])

Number of elements	Degree of polynomial		
	6.1.1	7.2.2	8.3.3
4	27.726 (33) 9,2 %	27.663 (49) 8,9 %	27.900 (65) 9,8 %
9	26.219 (76) 3,2 %	25.911 (112) 2,0 %	26.108 (148) 2,8 %
16	25.749 (137) 1,37 %	25.640 (201) 0,95 %	25.711 (265) 1,22 %

displacement formulation as used here have shown, that the displacement models converged partly from below and partly from above toward the exact solution. In that paper the influence on the convergence behaviour of natural frequencies dependent on the degree of polynomial within the element by fixing the degrees at the boundaries was also examined. Based on these investigations of the diverse hybrid displacement models the 6.1.1, 7.2.2 and 8.3.3 model have been chosen here for stability analysis.

In Table 5 the results obtained from the 7.2.2 model are also compared to those given by Allman[4], Clough and Felippa[12], Anderson *et al.*[13] using triangular elements and Kapur and Hartz[14], Dawe[15] and Carson and Newton[16] using rectangular elements. With respect to the number of elements the results obtained from the present hybrid displacement model are, for the cases given, superior to the other excepting to those given by Carson and Newton which are partly more accurate than those obtained from the 7.2.2 model. Examination of Table 5 reveals that for those two problems the 7.2.2 hybrid displacement model allows in general the use of a much coarser grid to obtain the same accuracy. With regard to the total number of degrees-of-freedom of the entire structure (given in paranthesis where the degrees-of-freedom from the boundary conditions are involved) the results given by Carson and Newton converge

Table 5. Critical loads of simply supported plate under uniform plane stresses

Loading case	Mesh size (over the whole plate)	Triangular elements			Rectangular elements			
		Allman	Clough and Felippa	Anderson et al.	Kapur and Hartz	Dawe	Carson and Newton	Present analysis (7.2.2 model)
Uniform uni-axial compression	2 x 2			3.22 (27)			4.015 (36)	4.018 (81)
	3 x 3			-	3.645 (48)		4.003 (64)	4.004 (160)
	4 x 4	4.031 (75)	4.126 (75)	3.72 (75)	3.77 (75)	3.978 (75)	4.001 (100)	-
	8 x 8	4.006 (243)	4.031 (243)	3.94 (243)	3.933 (243)	3.993 (243)	4.000 (324)	-
Uniform shear	2 x 2						10.016 (36)	10.289 (81)
	3 x 3						9.577 (64)	9.505 (160)
	4 x 4	10.131 (75)					9.418 (100)	9.383 (265)
	8 x 8	9.468 (243)						

more rapidly than those obtained from the 7.2.2 model. In all other cases the results from the present 7.2.2 model are more accurate.

In connection with the convergence behaviour of the buckling coefficient the matrix power series expansion should be considered. The essential step for deriving the elastic stiffness and geometric stiffness matrices is the expansion of a matrix in series form and the truncation of this series. The series is convergent only if all the eigenvalues of the matrix $[B]_E[H]_E = [R]_E$ are less than unity. Denoting the eigenvalues of

$$[R]_E\{x\}_E + \mu[I]_E\{x\}_E = 0 \qquad \mu_0 < \mu_1 < \cdots < \mu_n. \tag{61}$$

By means of the eigenvectors of eqn (61) as transformation matrix $[X]_E$ the expression

$$[[I]_E + \lambda[R]_E]^{-1} \tag{62}$$

in eqn (37) can be transformed to

$$[X]_E[[I]_E + \lambda[\mu]_E]^{-1}[X]_E^T \tag{63}$$

where $[\mu]_E$ is the diagonal matrix of egenvalues μ_i. The series is convergent only if

$$|\lambda\mu_0| \leq \|\lambda\mu_1| \leq \cdots \leq |\lambda\mu_n| < 1.$$

Thus the expansion is convergent for all eigenvalues λ_i which are less than $1/\mu_m$ and we can expect to obtain an approximation to those values by solving the linear eigenvalue problem of the entire structure. Since the binomial expansion has been truncated after the first term, we cannot expect an eigenvalue λ_i of the approximate solution of the discretized element equations to be exactly equal to an eigenvalue of the exact solution of the discretized element equations. The accuracy of the approximation to an eigenvalue will depend upon the value of the product $\lambda_i\mu_n$. To improve the accuracy, the value of μ_n has to be decreased. Numerical investigation of the values μ_n has shown that this can be obtained by decreasing the element size. This means it will be possible to get convergence of the buckling coefficient if the number of elements increases.

CONCLUSION

Based on a generalized potential energy principle the element system of equations has been derived. The exact solution of this system led to the (generalized) stiffness matrix whereas the approximate solution resulted in a formulation using the elastic stiffness and geometric stiffness matrix or in the case of buckling problems in a linear eigenvalue problem. The method has been applied to the computation of the buckling loads of plate bending problems. For simplicity, a rectangular element has been chosen to demonstrate the method. But it should be recognized that the method may be extended to more complicated continuous models, where the geometry and the degrees-of-freedom are difficult to interpret mechanically. The method outlined in the former section permits the derivation of elastic stiffness and geometric stiffness matrices where the number of degrees-of-freedom at the element boundaries contrary to the usual way in the finite element formulation (apart from the necessary condition that the number in the interior should not be smaller than at the element boundaries). This advantage permits an easier formulation of complicated finite element models such as shells etc., in dynamic and stability analysis.

REFERENCES

1. T. H. H. Pian, Derivation of element stiffness matrices by assuming stress distribution. *AIAA J.* **2**, 1333–1336 (1964).
2. P. Tong, New displacement hybrid finite element models for solid continua. *Int. J. Num. Meth. Engng* **2**, 73–83 (1970).
3. R. E. Jones, A generalization of the direct-stiffness method of structural analysis. *AIAA J.* **2**, 821–826 (1964).
4. S. J. Allman, Calculation of the elastic buckling loads of thin flat reinforced plates using triangular finite elements. *Int. J. Num. Meth. Engng* **9**, 415–432 (1975).
5. F. Kikuchi and Y. Ando, Some finite element solutions for plate bending problems by simplified hybrid displacement method. *Nuclear Engng and Design* **23**, 155–178 (1972).
6. P. Tong, S. T. Mau and T. H. H. Pian, Derivation of geometric stiffness and mass matrices for finite element hybrid models, *Int. J. Solids Structures* **10**, 919–932 (1974).
7. K. Brandt, Finite Elemente auf der Grundlage verallgemeinerter Variationsprinzipien zur Berechnung von Eigenfrequenzen mit einer Anwengung auf elastische Platten, Dissertation 1975, TH Darinstadt.
8. B. C. Greene,R. E. Jones, R. W. McLay and D. R. Strome, Generalized variational principles in the finite element method, *AIAA J.* **7**, 1254–1260 (1969).
9. T. H. Pian and P. Tong, The basis of finite element methods for solid continua, *Int J. Num. Meth. Engng* **1**, 3–28 (1968).
10. M. Link, Finite Elements nach Theorie II. Ordnung auf der Grundlage eines erweiterten variationsprinzips mit Anwendung auf die Stabilitäts-und Spannungsberechnung von einfachsymmetrischen *I*-Trägenrn unter Beruck-sichtigung der Querschnittsverformung, Dissertation 1972, Th Damstadt.
11. K. Klöppel and J. Scheer, *Beulwerte ausgesteifter Rechteckplatten.* Wilhelm Ernst and Sohn, Berlin (1960).
12. R. W. Clough and C. A. Felippa, A refined quadrilateral element for analysis of plate bending. Proceedings of the Second Conference on Matrix Methods in Structural Mechanics, AFFDL-TR-68-150, pp. 399–440 (Dec. 1969).
13. R. G. Anderson, B. M. Irons and O. C. Zienkiewicz, Vibration and stability of plates using finite elements, *Int. J. Solids and Structures* **4**, 1031–1055 (1968).
14. K. K. Kapur and B. J. Hartz, Stability of plates using the finite element method. *J. Engng Mech. Div., Proc. ASCE* **92**, 177–195 (1966).
15. D. J. Dawe, Application of the discrete element method to the buckling analysis of rectangular plates under arbitrary membrane loading. *Aero. Quart.*, Vol. XX, 114–128 (1969).
16. W. G. Carson and R. E. Newton, Plate buckling analysis using a fully compatible finite element. *AIAA J.* **7**, 527–529 (1969).